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Circular flow on signed graphs ☆

Andre Raspaud [a], Xuding Zhu [b]

[a] LaBRI, Universite Bordeaux I, France
[b] Department of Mathematics, Zhejiang Normal University, Jinhua, China

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ABSTRACT

The circular flow number $\Phi_c(G,\sigma)$ of a signed graph (G,σ) is the minimum r for which an orientation of (G,σ) admits a circular r-flow. We prove that the circular flow number of a signed graph (G,σ) is equal to the minimum imbalance ratio of an orientation of (G,σ). We then use this result to prove that if G is 4-edge-connected and (G,σ) has a nowhere zero flow, then $\Phi_c(G,\sigma)$ (as well as $\Phi(G,\sigma)$) is at most 4. If G is 6-edge-connected and (G,σ) has a nowhere zero flow, then $\Phi_c(G,\sigma)$ is strictly less than 4.

1. Introduction

All graphs in this paper are finite and loopless, but may have parallel edges. In a graph G, an edge $e = xy$ is viewed as two *half edges*: one half edge incident with x, and the other half edge incident with y. Denote by $E(G)$ the set of all edges of G, and by $H(G)$ the set of all half edges of G. If there is no confusion, $E(G)$ and $H(G)$ are abbreviated to E and H, respectively. For $h \in H$, let e_h be the edge containing h, let v_h be the vertex incident with h, and let $\bar{h}$ be the other half edge of e_h. If $e = xy \in E$, then we let h_e^x be the half edge of e incident with x. For a vertex v, $H_G(v)$ (abbreviated $H(v)$) is the set of half edges incident with v, and $E_G(v)$ (abbreviated $E(v)$) is the set of edges incident with v.

Suppose G is a graph and $\sigma : E(G) \to \{1, -1\}$ is a mapping. Then the pair (G,σ) is called a *signed graph*. An edge e is called a *positive edge* (or a *negative edge*) if $\sigma(e) = 1$ (or $\sigma(e) = -1$). For a subset E' of edges of G, let $\sigma(E') = \prod_{e \in E'} \sigma(e)$. Given a signed graph (G,σ), an orientation of (G,σ) is a mapping $\tau : H(G) \to \{1, -1\}$ such that for each edge e, if h, $\bar{h}$ are the two half edges of e, then $\tau(h)\tau(\bar{h}) = -\sigma(e)$. We view τ as an assignment of directions to the half edges of G. If $\tau(h) = 1$, then the half edge h is oriented away from v_h; if $\tau(h) = -1$, then the half edge is oriented towards v_h. The pair (G,τ) is called a *bidirected graph*. The signed graph (G,σ) is called the *underlying signed graph* of

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E-mail address: zhu@math.nsysu.edu.tw (X. Zhu).

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(G,τ), and the mapping σ is called the *signature* of τ. Observe that given a bidirected graph (G,τ), its underlying signed graph is uniquely determined. On the other hand, a signed graph (G,σ) have many distinct orientations. An edge e is called a positive edge or negative edge in a bidirected graph (G,τ) if it is a positive or negative edge in its underlying signed graph.

If all the edges of (G,σ) are positive, then (G,τ) is an *orientation* of G which assigns to each edge a direction: For an edge $e=xy$, if $\tau(h_e^x)=1$ and $\tau(h_e^y)=-1$, then the edge e is oriented from x to y. A bidirected graph (G,τ) with all edges positive is also called a *directed graph*; and a signed graph (G,σ) with all edges positive is a graph. In this sense, the class of graphs is a subclass of signed graphs, and the class of directed graphs is a subclass of bidirected graphs.

If $e=xy$ is an edge of a bidirected graph (G,τ) for which $\tau(h_e^x)=\tau(h_e^y)=1$, then the edge is oriented away from both x and y. If $\tau(h_e^x)=\tau(h_e^y)=-1$, then the edge is oriented towards both x and y. This may seem a little strange. However, such a bi-orientation arose naturally when one considers surface dual of oriented graphs embedded in non-orientable surfaces. Suppose G is a directed graph embedded on a surface Σ. The *surface dual* G^* of G has vertices all the faces of G, and each edge e of G corresponds to an edge e^* of G^* connecting the two faces incident to e in G. An orientation of G^* can be obtained as follows: For each face f of G, we assign a direction of traversal of the boundary of f as the positive direction of f. An edge e^* is oriented towards f (respectively, away from f) if the direction of the corresponding edge e in G agrees (respectively, disagrees) with the positive direction of f. In case Σ is an orientable surface, then the positive directions of the faces of G can be chosen in such a way that for each edge e of G, the direction of e agrees with one of the faces incident to e, and disagrees with the other face incident to e. In this case, the orientation of G^* defined above results in a directed graph. In case Σ is non-orientable, then the positive directions of the faces of G cannot be chosen consistently, and the orientation of G^* defined above results in a bidirected graph.

An important concept associated with directed graphs is nowhere zero k-flow, which is naturally extended to bidirected graphs [1]. Suppose (G,τ) is a bidirected graph. For a mapping $f: E \to \mathbb{R}$, the *boundary* of f is the map $\partial f: V(G) \to \mathbb{R}$ defined as

$$\partial f(v) = \sum_{h \in H(v)} \tau(h) f(e_h)$$

for each vertex v. If $\partial f = 0$ then f is a *flow* in (G,τ). The *support* of a flow f in (G,τ) is the set $supp(f) = \{e\colon\ f(e) \neq 0\}$. If f is an integer flow (i.e., $f(e)$ is an integer for each e) in (G,τ) and $1 \leqslant |f(e)| \leqslant k-1$ for each edge e, then f is a *nowhere zero k-flow* in (G,τ). The problem of interest is whether a given bidirected graph (or a directed graph) admits a nowhere zero k-flow.

For a bidirected graph (G,τ), the existence or non-existence of a nowhere zero k-flow is determined by the signature of τ: if bidirected graphs (G,τ) and (G,τ') have the same signature, then (G,τ) has a nowhere zero k-flow if and only if (G,τ') has a nowhere zero k-flow. Indeed, if f is a flow in (G,τ), then f' defined as $f'(e) = f(e)\tau(h_e^x)\tau'(h_e^x)$ (for $e=xy$) is a flow in (G,τ'). We say a signed graph (G,σ) admits a nowhere zero k-flow if an orientation (G,τ) of (G,σ) (and hence every orientation of (G,σ)) admits a nowhere zero k-flow. For a signed graph (G,σ), the *flow number* $\Phi(G,\sigma)$ is defined as

$$\Phi(G,\sigma) = \min\bigl\{k\colon\ (G,\sigma) \text{ admits a nowhere zero } k\text{-flow}\bigr\}.$$

In case (G,σ) does not admit a nowhere zero k-flow for any k, then let $\Phi(G,\sigma)=\infty$.

The study of flow number of graphs is an important and active branch of graph theory. Most of the research in this area are motivated by the following Tutte's flow conjectures:

(1) Every bridgeless graph has a nowhere zero 5-flow.
(2) Every bridgeless graph without 3-edge cut has a nowhere zero 3-flow.
(3) Every bridgeless graph containing no Petersen minor has a nowhere zero 4-flow.

Many interesting results are obtained in the study of these conjectures, although all the three conjectures are still open.

Nowhere zero k-flow in signed graphs was first studied by Bouchet [1]. It was proved in [1] that if a signed graph G admits a nowhere zero k-flow for some integer k, then it admits a nowhere zero 216-flow. Bouchet then proposed the following conjecture:

Conjecture 1. *If a signed graph G admits a nowhere zero k-flow for some k, then it admits a nowhere zero 6-flow.*

Conjecture 1 remains open, although there are improvements on Bouchet's result. It was proved in [17] that if a signed graph (G,σ) admits a nowhere zero k-flow for some k, then it admits a nowhere zero 30-flow. It was proved in [7] that if G is 4-edge-connected and the signed graph (G,σ) admits a nowhere zero k-flow for some integer k, then (G,σ) admits a nowhere zero 18-flow. Recently Xu and Zhang proved that if G is 6-edge-connected and the signed graph (G,σ) admits a nowhere k-flow for some integer k, then (G,σ) admits a nowhere zero 6-flow.

Nowhere zero flow of directed graphs is extended to circular r-flows of directed graphs [5,15,16], which can also be defined for bidirected graphs. Suppose (G,τ) is a bidirected graph and $r \geqslant 2$ is a real number. A mapping $f: E \to \mathbb{R}$ is called a *circular r-flow* in (G,τ) if the boundary of f is zero (i.e., for each vertex v, $\partial f(v) = \sum_{h \in H(v)} \tau(h) f(e_h) = 0$), and $1 \leqslant |f(e)| \leqslant r-1$ for each edge e. Similarly the existence or non-existence of a circular r-flow of a bidirected graph is determined by its underlying signed graph. We say a signed graph (G,σ) admits a circular r-flow if an orientation of (G,σ) (and hence every orientation of (G,σ)) admits a circular r-flow.

The *circular flow number* $\Phi_c(G,\sigma)$ of a signed graph (G,σ) is defined as

$$\Phi_c(G,\sigma) = \inf\bigl\{r\colon\ (G,\sigma) \text{ admits a circular } r\text{-flow}\bigr\}.$$

In case (G,σ) does not admit a circular r-flow for any r, then let $\Phi_c(G,\sigma) = \infty$.

It follows from the definition that for any positive integer k, a nowhere zero k-flow in a bidirected graph (G,τ) is also a circular k-flow in (G,τ). Therefore for any signed graph (G,σ),

$$\Phi_c(G,\sigma) \leqslant \Phi(G,\sigma).$$

We conjecture that $\Phi_c(G,\sigma) > \Phi(G,\sigma) - 1$ for every signed graph (G,σ). However, we are only able to prove that for any signed graph (G,σ),

$$\Phi(G,\sigma) \leqslant 2\bigl\lceil \Phi_c(G,\sigma)\bigr\rceil - 1.$$

Given an orientation (G,τ) of a signed graph (G,σ), we introduce the concept of imbalance ratio of (G,τ) (see Section 4 for the definition). Then we prove that

$$\Phi_c(G,\sigma) = \min\bigl\{r\colon\ (G,\sigma) \text{ has an orientation whose imbalance ratio is } r\bigr\}.$$

Then we study the circular flow number of highly edge-connected signed graphs and prove the following result.

Theorem 1. *Suppose (G,σ) is a signed graph and (G,σ) admits a nowhere zero k-flow for some integer k.*

1. *If G is 4-edge-connected, then $\Phi_c(G,\sigma) \leqslant \Phi(G,\sigma) \leqslant 4$.*
2. *If G is 6-edge-connected, then $\Phi_c(G,\sigma) < 4$.*

An unpublished manuscript [2] of M. DeVos contains a theorem which says that if G is 4-edge-connected and (G,σ) is a signed graph which admits a nowhere zero k-flow for some integer k, then (G,σ) admits a nowhere zero 4-flow. This theorem would imply the first part of Theorem 1. However, the proof in [2] contains an error. The proof presented here corrects that error. The second part is a generalization of a result of Galluccio and Goddyn [4], who proved that 6-edge-connected graphs G have $\Phi_c(G) < 4$.

2. Notation and preliminary results

Given a signed graph (G,σ) and a subset E' of edges of G, we also denote by E' (respectively, (E',σ)) the subgraph (respectively, the signed subgraph) of G induced by the edges in E'. If X is subset of $V(G)$, then $G[X]$ (respectively, $(G[X],\sigma)$) denote the subgraph (respectively, the signed subgraph) of G induced by X.

A *circuit* in a signed graph (G,σ) is a connected 2-regular subgraph of G. If C is a circuit, then (C,σ) is *balanced* (respectively, *unbalanced*) if it contains an even number (respectively, an odd number) of negative edges.

A signed graph (H,σ) is called a *barbell* if either

- H consists of two unbalanced circuits C_1, C_2 with $|V(C_1)\cap V(C_2)|=1$, or
- H consists of two vertex disjoint unbalanced circuits C_1, C_2 and a path P, which has one end in $V(C_1)$ and one end in $V(C_2)$ and has no interior vertices in $V(C_1)\cup V(C_2)$.

A signed graph (H,σ) is called a *signed circuit* if (H,σ) is either a balanced circuit or a barbell. A signed graph is *s-bridgeless* if every edge of G is contained in a signed circuit.

Suppose (H,σ) a signed circuit, and (H,τ) is an orientation of (H,σ). We define a *characteristic flow* f of (H,σ) as follows:

Assume (H,σ) is a balanced circuit, say $H=(v_0,e_0,v_1,e_1,\dots,v_{n-1},e_{n-1},v_0)$. Let h_i be the half edge of e_i incident to v_i. Let f be defined as $f(e_i)=\tau(h_i)\prod_{j=0}^{i-1}\sigma(e_j)$ for $i=1,2,\dots,n-1$ (by convention, $f(e_0)=\tau(h_0)$). Then it is easy to verify that f is a flow with support $E(H)$. Both f and $-f$ are called characteristic flows of (H,σ) (see Fig. 1).

Suppose (H,σ) is a barbell with two unbalanced circuits C_1, C_2 and a path P (possibly empty) connecting C_1 and C_2. Assume that

$$\begin{aligned}
&C_1=(v_0,e_0,v_1,e_1,\dots,v_{n-1},e_{n-1},v_0),\\
&C_2=\left(v'_0,e'_0,v'_1,e'_1,\dots,v'_{n'-1},e'_{n'-1},v'_0\right),\\
&P=\left(v_0=u_0,e''_0,u_1,e''_1,\dots,u_{t-1},e''_{t-1},v'_0=u_t\right).
\end{aligned}$$

In case P is empty, then $v_0=v'_0$. Let h_i be the half edge of e_i incident to v_i, h'_i be the half edge of e'_i incident to v'_i, and h''_i be the half edge of e''_i incident with u_i. Let f be defined as follows:

$$\begin{aligned}
&f(e_i)=\tau(h_i)\prod_{j=0}^{i-1}\sigma(e_j),\quad i=0,1,\dots,n-1,\\
&f\left(e''_i\right)=-2\tau\left(h''_i\right)\prod_{j=0}^{i-1}\sigma\left(e''_j\right),\quad i=0,1,\dots,t-1,\\
&f\left(e'_i\right)=\mu\tau\left(h'_i\right)\prod_{j=0}^{i-1}\sigma\left(e'_j\right),\quad i=0,1,\dots,n'-1,
\end{aligned}$$

where $\mu=\frac{1}{2}f(e''_{t-1})\tau(h''_{t-1})\sigma(e''_{t-1})$, or equivalently, $\mu=1$ or -1, depending on the flow from P to v'_0 is positive or negative. By convention, if $i=0$, then $\prod_{j=0}^{i-1}\sigma(e_j)=\prod_{j=0}^{i-1}\sigma(e''_j)=\prod_{j=0}^{i-1}\sigma(e'_j)=1$. Hence $f(e_0)=\tau(h_0)$, $f(e''_0)=-2\tau(h''_0)$ and $f(e'_0)=\mu\tau(h'_0)$. Again, it is easy to verify that f is a flow with support $E(H)$. Both f and $-f$ are called characteristic flows of (H,σ).

If each edge of (G,σ) is contained in a signed circuit, then appropriate linear combination of the characteristic flows of the signed circuits of (G,σ) will be a nowhere zero k-flow in (G,σ) for some k. Conversely, it is known [1] that if a signed graph (G,σ) has a nowhere zero k-flow for some integer k then every edge of G is contained in some signed circuit, i.e., (G,σ) is s-bridgeless. So Bouchet conjecture is equivalent to say that every s-bridgeless signed graph admits a nowhere zero 6-flow.

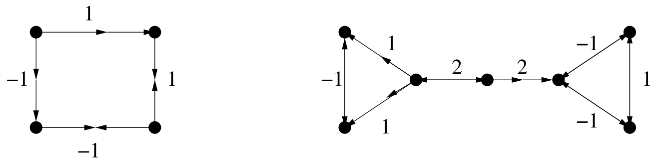

Fig. 1. Example of signed circuits and characteristic flows.

Suppose (G,σ) is a signed graph and $v \in V(G)$ is a vertex of G. Let

$$\sigma'(e) = \begin{cases} -\sigma(e), & \text{if } e \in E(v), \\ \sigma(e), & \text{otherwise.} \end{cases}$$

Then we say σ' is obtained from σ by a *switch* at v. Two signed graphs are *equivalent* if one can be obtained from the other by a sequence of switches. Assume (G,σ') is obtained from (G,σ) by a switch at a vertex v and (G,τ) is an orientation of (G,σ). Let $\tau' : H(G) \to \{-1,1\}$ be defined as

$$\tau'(h) = \begin{cases} -\tau(h), & \text{if } h \in H_G(v), \\ \tau(h), & \text{otherwise.} \end{cases}$$

Then τ' is an orientation of (G,σ'). Moreover, if f is a flow in (G,τ), then f is also a flow in (G,τ'). So equivalent signed graphs have the same flow number and circular flow number.

A signed graph (G,σ) is called a *balanced signed graph* if each circuit of (G,σ) is balanced. If (G,σ) is a balanced signed graph, then there is a mapping $c : V(G) \to \{1,2\}$ such that the following holds: If $e = xy$ is a negative edge, then $c(x) \neq c(y)$; if $e = xy$ is a positive edge, then $c(x) = c(y)$. In other words, there is a subset Y (namely $Y = c^{-1}(1)$) of $V(G)$ such that all edges between Y and $\bar{Y}$ are negative and all other edges are positive. By switching at all the vertices in Y, we obtain a signed graph (G,σ') in which all edges are positive, i.e., (G,σ') is a graph. It is easy to see that the converse is also true. So a signed graph (G,σ) is balanced if and only if it is equivalent to a graph, i.e., a signed graph in which all edges are positive.

The following lemma is a characterization of s-bridgeless signed graphs (the only if part of the lemma was proved in [1] and the if part is easy (cf. [2,9])).

Lemma 1. *A connected signed graph (G,σ) is s-bridgeless (and hence admits a nowhere zero k-flow for some integer k) if and only if the following hold:*

- *(G,σ) is not equivalent to a signed graph (G,σ') with exactly one negative edge.*
- *If e is a cut-edge of G and H is a component of $G - e$, then (H,σ) is not balanced.*

Another important concept in the study of flows in bidirected graphs is the matroid of the underlying signed graphs. Suppose (G,τ) is a bidirected graph. Let $B = [b_{ve}]$ be the incidence matrix of (G,τ), where $b_{ve} = \tau(h_e^v)$. The matroid $M(G,\sigma)$ of its underlying signed graph (G,σ) is the matroid of the linear dependencies on the columns of B. The matroid $M(G,\sigma)$ was first studied by Zaslavsky [13,14]. We shall only need the properties of the matroid described in the following two theorems, each of which can also be treated as a definition of the matroid.

Theorem 2 *(Zaslavsky). Let (G,σ) be a signed graph, a set B of edges of $M(G,\sigma)$ is a circuit if (B,σ) is a signed circuit, i.e., (B,σ) is either a balanced circuit C, or a barbell.*

Theorem 3 *(Zaslavsky). Given a connected signed graph (G,σ). If (G,σ) is balanced, then B is a base of $M(G,\sigma)$ if and only if B is a signed spanning tree. If (G,σ) is not balanced, then B is a base of $M(G,\sigma)$ if and only if each component of B contains a unique circuit and the circuit is unbalanced.*

Suppose (G,σ) is a connected signed graph. We say a base B of $M(G,\sigma)$ is *connected* if B has only one component. If (G,σ) is balanced, then every base of $M(G,\sigma)$ is connected. If (G,σ) is unbalanced, then a base B is connected if B is a spanning tree with one extra edge and contains a unique unbalanced circuit.

3. Relation between $\Phi_c(G,\sigma)$ and $\Phi(G,\sigma)$

It follows from the definition that $\Phi_c(G,\sigma) \leqslant \Phi(G,\sigma)$ for any signed graph (G,σ). If (G,σ) is balanced (i.e., if (G,σ) is equivalent to a graph), then we know that $\Phi(G) = \lceil \Phi_c(G) \rceil$. For arbitrary signed graphs G, it is unknown if the equality $\Phi(G) = \lceil \Phi_c(G) \rceil$ still holds. In this section, we prove the following result.

Theorem 4. *For any s-bridgeless signed graph G, $\Phi(G) \leqslant 2\lceil \Phi_c(G) \rceil - 1$.*

Proof. Let $k = \lceil \Phi_c(G) \rceil$. Let (G,τ) be an orientation of (G,σ). By definition, (G,τ) has a circular k-flow (if $r' \geqslant r$ then a circular r-flow in (G,τ) is also a circular r'-flow in (G,τ)). Given a circular k-flow f of (G,τ), let

$$E(f) = \{e \in E(G):\ f(e) \text{ is not an integer}\}.$$

Choose a circular k-flow f of (G,τ) for which $E(f)$ has minimum cardinality.

Since f is a flow, for each vertex v, $\sum_{h \in H(v), e_h \in E(f)} \tau(h) f(e_h)$ is an integer. In particular, v is incident to either 0 edges of $E(f)$ or at least two edges of $E(f)$. If $(E(f),\sigma)$ contains a signed circuit (H,σ). Then let g be a characteristic flow of (H,τ). Let $\delta > 0$ be the maximum real number such that for each edge e of H,

$$\lfloor f(e) \rfloor \leqslant f(e) - \delta g(e),\ f(e) + \delta g(e) \leqslant \lceil f(e) \rceil.$$

Then both $f + \delta g$, $f - \delta g$ are circular k-flows in (G,τ) and either $E(f+\delta g)$ or $E(f-\delta g)$ is a proper subset of $E(f)$, in contrary to the choice of f.

So $E(f)$ contains no signed circuits. As observed above, no vertex of G is incident to exactly one edge of $E(f)$. Assume there is a vertex incident with at least three edges of $E(f)$. Then $E(f)$ induces a graph of minimum degree at least 2 in which one vertex has degree at least 3. So $E(f)$ contains two circuits that either intersect each other or are connected by a path. If two circuits intersect in at least two vertices, then one can easily obtain a balanced circuit (and hence a signed circuit) from the union of the two circuits. If $E(f)$ contains two circuits that intersect at one vertex or are connected by a path, then $E(f)$ contains either a balanced circuit or a barbel and hence a signed circuit, a contradiction. Thus each vertex of G is incident to 0 or 2 edges of $E(f)$. Therefore each non-empty component of $(E(f),\sigma)$ is an unbalanced circuit.

Let (C,σ) be a component of $(E(f),\sigma)$ which is an unbalanced circuit. Assume $C = (v_0, e_0, v_1, e_1, \ldots, v_{n-1}, e_{n-1}, v_0)$. Let h_i be the half edge of e_i incident with v_i. For each v_i,

$$f(e_i)\tau(h_i) + f(e_{i-1})\tau(\bar{h}_{i-1}) \cong \sum_{h \in H(v_i), e_h \in E(f)} \tau(h) f(e_h) \pmod 1 \cong 0 \pmod 1.$$

In other words, if $f(e_i)\tau(h_i) \cong \delta \pmod 1$, then $f(e_{i-1})\tau(\bar{h}_{i-1}) \cong -\delta \pmod 1$. If e_{i-1} is a positive edge, then $f(e_{i-1})\tau(h_{i-1}) \cong \delta \pmod 1$; if e_{i-1} is a negative edge, then $f(e_{i-1})\tau(h_{i-1}) \cong -\delta \pmod 1$. Without loss of generality, we assume that $f(e_0)\tau(h_0) \cong \delta \pmod 1$. Since C is unbalanced, so if we start from e_0 and go a full round along the cycle C, $f(e_i)\tau(h_i)$ will change signs an odd number of times and hence we arrived at the conclusion that $f(e_0)\tau(h_0) \cong -\delta \pmod 1$. So $\delta \cong -\delta \pmod 1$, i.e., $\delta = 1/2$. Therefore for any edge e of G, $2f(e)$ is an integer, i.e., $2f(e)$ is an integer flow. As $1 \leqslant |2f(e)| \leqslant 2k-2$ for each edge e, we conclude that $2f$ is a nowhere zero $(2k-1)$-flow. □

4. Imbalance ratio of orientations

Suppose (G,τ) is a directed graph and X is a subset of $V(G)$. Denote by $\partial(X)$ the set of edges with exactly one end vertex in X, by $\partial^+(X)$ the set of edges in $\partial(X)$ oriented from X to $\bar{X}$, and by

$\partial^-(X)$ the set of edges in $\partial(X)$ oriented from $\bar{X}$ to X. It follows from Hoffman's theorem [5,6] that a graph G has a circular r-flow if and only if G has an orientation (G,τ) such that for each subset X of $V(G)$, $|\partial^+(X)| \leqslant (r-1)|\partial^-(X)|$ and $|\partial^-(X)| \leqslant (r-1)|\partial^+(X)|$.

This section generalizes this result to signed graphs. First we need the corresponding notation of X, $\partial(X)$, $\partial^+(X)$ and $\partial^-(X)$ for signed graphs.

Suppose (G,σ) is a signed graph. A *signed subset* of $V(G)$ is a pair (X,θ), where X is a subset of $V(G)$ and $\theta : X \to \{1,-1\}$ is a mapping. In other words, a signed subset is a subset X together with a partition $X = X^+ \cup X^-$, where $X^+ = \{x \in X\colon\ \theta(x) = 1\}$ and $X^- = \{x \in X\colon\ \theta(x) = -1\}$. If the mapping θ is clear from the context (or is insignificant), we may write X for (X,θ).

Given a signed subset (X,θ) of (G,σ), let

$$\partial_{G,\sigma}(X,\theta) = \bigl\{h \in H(G)\colon\ v_h \in X \text{ and } v_{\bar{h}} \notin X \text{ or } v_h \in X,\ v_{\bar{h}} \in X \text{ and } \theta(v_h)\theta(v_{\bar{h}})\sigma(e_h) = -1\bigr\},$$
$$E\bigl(\partial_{G,\sigma}(X,\theta)\bigr) = \bigl\{e_h\colon\ h \in \partial_{G,\sigma}(X,\theta)\bigr\}.$$

Lemma 2. *Suppose (G,σ) is a signed graph and (X,θ) is a signed subset of $V(G)$. If $\partial_{G,\sigma}(X,\theta) \neq \emptyset$, then $E(\partial_{G,\sigma}(X,\theta))$ intersects every base of $M(G,\sigma)$.*

Proof. Let B be a base of $M(G,\sigma)$. If there is a component (T,σ) of (B,σ) which contains both vertices of X and vertices of $\bar{X}$, then T contains an edge connecting a vertex of X and a vertex of $\bar{X}$. Therefore $B \cap E(\partial_{G,\sigma}(X,\theta)) \neq \emptyset$.

Assume for each component of (B,σ), its vertex set is either contained in X or disjoint from X. Let T be a component with $V(T) \subseteq X$.

If (G,σ) is balanced, then B is a spanning tree, and hence $V(T) = V(G)$. So $X = V(G)$. As (G,σ) is balanced, there is a subset Y of $V(G)$ such that all the edges between Y and $\bar{Y}$ are negative, and all the other edges are positive. Let $Z = (Y \cap X^-) \cup (X^+ - Y)$. Then it is straightforward to verify that $E(\partial_{G,\sigma}(X,\theta))$ contains all edges between Z and $V \setminus Z$. Moreover, if $Z = \emptyset$ or $Z = V$, then $\partial_{G,\sigma}(X,\theta) = \emptyset$, in contrary to our assumption. Therefore $E(\partial_{G,\sigma}(X,\theta)) \cap B \neq \emptyset$.

Assume (G,σ) is unbalanced. Then T contains a unique unbalanced circuit C. By contracting all positive edges of C, we obtain an odd circuit. Therefore, either there is a positive edge $e = xy$ of C such that $\theta(x)\theta(y) = -1$ or there is a negative edge $e = xy$ of C such that $\theta(x)\theta(y) = 1$. In any case, $e \in E(\partial_{G,\sigma}(X,\theta))$ and hence $E(\partial_{G,\sigma}(X,\theta)) \cap B \neq \emptyset$. This completes the proof of the lemma. $\square$

Recall that a cocircuit of a matroid M is a minimal subset of M which intersects each base of M. Lemma 2 is equivalent to say that $E(\partial_{G,\sigma}(X,\theta))$ contains a cocircuit of the matroid $M(G,\sigma)$. The set $E(\partial_{G,\sigma}(X,\theta))$ itself is not necessarily a cocircuit of $M(G,\sigma)$, but instead the union of cocircuits of $M(G,\sigma)$.

Suppose (G,τ) is an orientation of a signed graph (G,σ) and (X,θ) is a signed subset of $V(G)$. Let

$$\partial^+_{G,\tau}(X,\theta) = \bigl\{h \in \partial_{G,\sigma}(X,\theta)\colon\ \theta(v_h)\tau(h) = 1\bigr\},$$
$$\partial^-_{G,\tau}(X,\theta) = \bigl\{h \in \partial_{G,\sigma}(X,\theta)\colon\ \theta(v_h)\tau(h) = -1\bigr\}.$$

The *imbalance ratio* of the bidirected graph (G,τ) is defined to be the maximum of $\frac{|\partial_{G,\sigma}(X,\theta)|}{\partial^+_{G,\tau}(X,\theta)}$ among all the signed subsets (X,θ) of $V(G)$.

If there is no confusion, we usually write $\partial^+ X$ for $\partial^+_{G,\tau}(X,\theta)$ and $\partial^- X$ for $\partial^-_{G,\tau}(X,\theta)$.

Lemma 3. *A signed graph (G,σ) has a circular r-flow if and only if (G,σ) has an orientation (G,τ) such that for any signed subset (X,θ) of G for which $E(\partial_{G,\sigma}(X,\theta)) \neq \emptyset$,*

$$1/(r-1) \leqslant \bigl|\partial^+ X\bigr| / \bigl|\partial^- X\bigr| \leqslant r-1.$$

Proof. ($\Rightarrow$) We assume that (G,σ) has a circular r-flow. Then there is an orientation (G,τ) of (G,σ) which has a circular r-flow f with $f(e) > 0$ (and hence $1 \leqslant f(e) \leqslant r-1$) for each edge e. Let X be a signed subset of $V(G)$. Then

$$0 = \sum_{v\in X^+}\Bigl(\sum_{h\in H(v)} \tau(h) f(e_h)\Bigr)$$

and

$$0 = \sum_{v\in X^-}\Bigl(\sum_{h\in H(v)} \tau(h) f(e_h)\Bigr).$$

Hence

$$\begin{aligned}
0 &= \sum_{v\in X^+}\Bigl(\sum_{h\in E(v)} \tau(h) f(e_h)\Bigr) - \sum_{v\in X^-}\Bigl(\sum_{h\in E(v)} \tau(h) f(e_h)\Bigr)\\
&= \sum_{v\in X^+}\Bigl(\sum_{h\in E(v),\tau(h)=1} f(e_h)\Bigr) - \sum_{v\in X^+}\Bigl(\sum_{h\in E(v),\tau(h)=-1} f(e_h)\Bigr)\\
&\quad - \sum_{v\in X^-}\Bigl(\sum_{h\in E(v),\tau(h)=1} f(e_h)\Bigr) + \sum_{v\in X^-}\Bigl(\sum_{h\in E(v),\tau(h)=-1} f(e_h)\Bigr)\\
&= \sum_{h\in\partial^+X} f(e_h) - \sum_{h\in\partial^-X} f(e_h).
\end{aligned}$$

Since f is a circular r-flow we have

$$\bigl|\partial^+X\bigr| \leqslant \sum_{h\in\partial^+X} f(e_h) = \sum_{h\in\partial^-X} f(e_h) \leqslant (r-1)\bigl|\partial^-X\bigr|.$$

Similarly, we have

$$\bigl|\partial^-X\bigr| \leqslant (r-1)\bigl|\partial^+X\bigr|.$$

This gives the asked inequalities.

($\Leftarrow$) Let τ be an orientation of (G,σ) such that for any signed subset X of $V(G)$ we have:

$$1/(r-1) \leqslant \bigl|\partial^+X\bigr|/\bigl|\partial^-X\bigr| \leqslant r-1.$$

Let f be a flow on (G,τ) such that

1. $0 \leqslant f(e_h) \leqslant r-1$ for any $h \in H(G)$.
2. Subject to (1), $\sum_{h: f(e_h)<1}(1-f(e_h))$ is minimum.

Observe that a flow f satisfying (1) exists, because $f(e)=0$ for all edges e is such a flow. If $f(e) \geqslant 1$ for all edges e, then f is a circular r-flow and we are done. Assume this is not the case, i.e., there exists an edge e with $f(e) < 1$.

A signed circuit (H,σ) in (G,τ) is called *augmentable* with respect to f if the following hold: There is a characteristic flow g of (H,τ) such that (1) for any edge $e \in E(H)$, if $g(e) > 0$, then $f(e) < r-1$; (2) if $g(e) < 0$, then $f(e) > 1$; (3) there is an edge e for which $g(e) > 0$ and $f(e) < 1$.

Observe that if (G,τ) has an augmentable signed circuit (H,σ), then if $\delta > 0$ is sufficiently small, then $f(e) + \delta g(e) \leqslant r-1$ and $f(e) + \delta g(e) < 1$ only if $f(e) < 1$. We can define a new flow f': $f'(e) = f(e)$ for any $e \in E(G) \setminus E(H)$, and $f'(e) = f(e) + \delta g(e)$ for $e \in E(H)$. As there is an edge e for which $f(e) < 1$ and $g(e) > 0$, and hence $f'(e) = f(e) + \delta g(e) > f(e)$, this is in contrary to the choice of f. Thus we assume that (G,τ) has no augmentable signed circuits.

In the remainder of the proof, let $e^* = xy$ be a fixed edge of G with $f(e^*) < 1$. We may assume that $\sigma(e^*) = 1$ (the case $\sigma(e^*) = -1$ is considered in a similar way). Moreover, without loss of generality, we assume that $\tau(h_{e^*}^y) = -1$.

Suppose $P = (v_0, v_1, \dots, v_{k-1}, v_k)$ is a walk in $G - e^*$. Let $e_j = v_j v_{j+1}$ for $j = 0, 1, \dots, k-1$, and let $\theta : V(P) \to \{1, -1\}$ satisfies $\theta(v_{j+1}) = \theta(v_j)\sigma(e_j)$ for $j = 0, 1, \dots, k-2$. If

- the vertices $v_0, v_1, \dots, v_{k-1}$ are distinct,
- and $v_k = v_{k'}$ for some $k' \leqslant k-2$,
- and $\theta(v_{k-1})\theta(v_k)\sigma(e_{k-1}) = -1$,

then we say P is a *tadpole starting from* v_0.

Claim 1. *Assume $P = (v_0, v_1, \dots, v_{k-1}, v_k)$ is a tadpole, where $v_k = v_{k'}$ for some $k' \leqslant k-2$ and $e_j = v_j v_{j+1}$. For the mapping $\phi : E(G) \to \{-2, -1, 0, 1, 2\}$ defined as*

$$\phi(e) = \begin{cases} 2\theta(v_j)\tau(h_{e_j}^{v_j}), & \text{if } e = e_j \text{ for } j = 0, 1, \dots, k'-1, \\ \theta(v_j)\tau\big(h_{e_j}^{v_j}\big), & \text{if } e = e_j \text{ for } j = k', k'+1, \dots, k-1, \\ 0, & \text{otherwise,} \end{cases}$$

we have $\partial\phi(v_0) = 2\theta(v_0)$ and $\partial\phi(u) = 0$ for $u \neq v_0$.

This claim can be verified directly from the definition. For example, by definition,

$$\partial\phi(v_0) = \sum_{h \in H(v_0)} \tau(h)\phi(e_h) = \tau\big(h_{e_0}^{v_0}\big)2\theta(v_0)\tau\big(h_{e_0}^{v_0}\big) = 2\theta(v_0)$$

and

$$\begin{aligned}\partial\phi(v_{k'}) &= \sum_{h \in H(v_0)} \tau(h)\phi(e_h) \\ &= \tau(h_{e_{k'-1}^{v_{k'}}})\phi(e_{k'-1}) + \tau\big(h_{e_{k'}}^{v_{k'}}\big)\phi(e_{k'}) + \tau\big(h_{e_{k-1}}^{v_k}\big)\phi(e_{k-1}).\end{aligned}$$

Since

$$\begin{aligned}&\phi(e_{k'-1}) = 2\theta(v_{k'-1})\tau\big(h_{e_{k'-1}}^{v_{k'-1}}\big), \\ &\sigma(e_{k'-1}) = -\tau\big(h_{e_{k'-1}}^{v_{k'-1}}\big)\tau\big(h_{e_{k'-1}}^{v_{k'}}\big), \\ &\theta(v_{k'}) = \theta(v_{k'-1})\sigma(e_{k'-1}),\end{aligned}$$

it follows that $\tau(h_{e_{k'-1}}^{v_{k'}})\phi(e_{k'-1}) = -2\theta(v_{k'})$. Similarly, it can be shown that $\tau(h_{e_{k'}}^{v_{k'}})\phi(e_{k'}) = \tau(h_{e_{k-1}}^{v_k})\phi(e_{k-1}) = \theta(v_{k'})$. Hence $\partial\phi(v_{k'}) = 0$.

Suppose $P = (v_0, v_1, \dots, v_{k-1}, v_k)$ is a tadpole. If $\theta(v_0) = 1$ then it is called a *positive tadpole*, otherwise it is called a *negative tadpole*. If the following hold:

- If $\theta(v_j)\tau(h_{e_j}^{v_j}) = 1$ then $f(e_j) < r-1$, if $\theta(v_j)\tau(h_{e_j}^{v_j}) = -1$ then $f(e_j) > 1$.

Then we say the tadpole P is *augmentable* (with respect to f).

Assume there is a positive tadpole $P = (v_0, v_1, \dots, v_{k-1}, v_k)$, with $v_k = v_{k'}$ for some $k' \leqslant k-2$, starting from y and a negative tadpole $P' = (v'_0, v'_1, \dots, v'_{m-1}, v'_m)$, with $v'_m = v'_{m'}$ for some $m' \leqslant m-2$, starting from x, and both tadpoles are augmentable. The two tadpoles may have non-empty intersection. It is straightforward to verify that if P and P' have a vertex in common then the union $P \cup P'$ contains an augmentable balanced circuit. If P and P' have no vertex in common, then $P \cup P'$ is an augmentable barbell. In any case, we obtain an augmentable signed circuits of (G, τ). This is in contrary to our assumption.

So either (G, τ) has no positive augmentable tadpole starting from y or (G, τ) has no negative augmentable tadpole starting from x. Without loss of generality, we assume that G has no positive augmentable tadpole starting from y.

We then recursively construct a signed subset (X, θ) of $V(G)$ by the following rules:

1. Initially $X = \{y\}$ and $\theta(y) = 1$.
2. Assume $e = wt$ is an edge of $G - e^*$ with $w \in X$ and $t \notin X$. If $\theta(w)\tau(h_e^w) = 1$ and $f(e) < r - 1$, then we add t to X. If $\theta(w)\tau(h_e^w) = -1$ and $f(e) > 1$, then we add t to X. In any case, we let $\theta(t) = \theta(w)\sigma(e)$.

The above process will terminate as G is a finite graph, and we obtain a signed subset X of $V(G)$. It follows from the definition that for each vertex $v \in X$, there is a y-v-path $P_v = (v_0, v_1, \dots, v_k)$ such that $v_0 = y$, $v_k = v$, and for $e_j = v_j v_{j+1}$, if $\theta(v_j)\tau(h_{e_j}^{v_j}) = 1$, then $f(e_j) < r - 1$; if $\theta(v_j)\tau(h_{e_j}^{v_j}) = -1$, then $f(e_j) > 1$.

If $x \in X$ and $\theta(x) = 1$, then $P_x + e^* = (v_0, v_1, \dots, v_k, v_0)$, with $v_0 = y$ and $v_k = x$, is an augmentable circuit with respect to f, in contrary to our assumption. Thus we assume that either $x \notin X$ or $x \in X$ but $\theta(x) = -1$. Therefore $e^* \in \partial^- X$.

If the following hold:

- $\forall h \in \partial^+ X$, $f(e_h) = r - 1$, and
- $\forall h \in \partial^- X$, $f(e_h) \leqslant 1$,

then, since $h_{e^*}^x \in \partial^- X$ and $f(e^*) < 1$, we have

$$(r-1)\left|\partial^+ X\right| = \sum_{h \in \partial^+ X} f(e_h) = \sum_{h \in \partial^- X} f(e_h) < \left|\partial^- X\right|.$$

This implies that

$$1/(r-1) > \left|\partial^+ X\right| / \left|\partial^- X\right|,$$

in contrary to our assumption.

Thus we may assume that there exists either an $h \in \partial^+ X$ with $f(e_h) < r - 1$ or an $h \in \partial^- X$ with $f(e_h) > 1$. Assume $e_h = wt$, and without loss of generality, assume that $w \in X$. By the construction, we should have put t to X. Thus both vertices w, t are in X. Thus by definition of ∂X, we have $\theta(w)\theta(t)\sigma(e_h) = -1$.

Let $P_w = (v_0, v_1, \dots, v_k)$ be the y-w-path and $P_t = (u_0, u_1, \dots, v_m)$ be the y-t-path defined as before. Thus $v_0 = u_0 = y$ and $v_k = w$, $u_m = t$. Let i be the largest index such that $v_i \in P_t$, say $v_i = u_{i'}$. Let T be the subgraph of G induced by the edges of P_w and the edges of $P'_t = (u_{i'}, u_{i'+1}, \dots, u_m)$. Then T is a tree. Other than v_0, T has two leaves w and t. Let P be obtained from T by adding the edge wt. We view P as a walk $(v_0, v_1, \dots, v_k, u_m, u_{m-1}, \dots, u_{i'+1}, u_{i'})$, where all the vertices $v_0, v_1, \dots, v_k, u_m, u_{m-1}, \dots, u_{i'+1}$ are distinct and $u_{i'} = v_i$. Since $\theta(w)\theta(t)\sigma(e_h) = -1$, by letting $\theta'(v_j) = \theta(v_j)$ and $\theta'(u_j) = -\theta(u_j)$, we can see that P is indeed a tadpole. Moreover, it follows from the construction that P is a positive tadpole, augmentable with respect to f. This is in contrary to our assumption. $\square$

Corollary 1. *For a signed graph (G, σ),*

$$\Phi_c(G, \sigma) = \min\left\{r\text{: } (G, \sigma) \text{ has an orientation whose imbalance ratio is } r\right\}.$$

5. Connected disjoint bases of $M(G, \sigma)$

It is proved in [8] by using the results of Tutte [11] and Nash-Williams [10] that if G is a $2k$-edge-connected graph, then G has k-edge disjoint spanning trees. This result is extended to matroid by

Edmonds [3]: A matroid M has k disjoint bases if and only if for every subset X of $E(M)$, $kr(X) + |E(M) - X| \geqslant kr(M)$.

We say a signed graph (G,σ) is *k-unbalanced* if for any balanced subgraph G' of G, $|E(G) - E(G')| \geqslant k$.

In this section, we shall prove that if G is $2k$-edge-connected and k-unbalanced, then $M(G,\sigma)$ has k disjoint connected bases.

Since G is $2k$-edge-connected, we know that G has k edge disjoint spanning trees. Each spanning tree of G is an independent set in $M(G,\sigma)$. A family $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ of disjoint independent set is called optimal, if each F_i contains a spanning tree of G and the total number of edges in the F_i's is maximum.

Suppose $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ is a family of optimal disjoint independent sets of $M(G,\sigma)$. We define a sequence of sets $E_0(\mathcal{F})$, $E_1(\mathcal{F})$, $\ldots$, as follows: Let

$$E_0(\mathcal{F}) = E(G) - \bigcup_{j=1}^{k} F_j.$$

Suppose $E_{j-1}(\mathcal{F})$ is defined. For each $e \in E_{j-1}(\mathcal{F})$, for each $F_i \in \mathcal{F}$, let

$$C(F_i,e) = \begin{cases} C, & \text{if } F_i + e \text{ contains a balanced cycle } C; \\ C_1 \cup C_2, & \text{if } F_i + e \text{ contains a barbell in which } C_1, C_2 \text{ are} \\ & \text{two unbalanced circuits;} \\ \{e\}, & \text{if } e \in F_i. \end{cases}$$

In this definition, the case $e \notin F_i$ and $F_i + e$ independent is not considered. As we shall prove in Lemma 4 below, this case will never happen.

Let

$$E_j(\mathcal{F}) = \bigcup_{e \in E_{j-1}(\mathcal{F}), F_i \in \mathcal{F}} C(F_i,e).$$

Note that for $j \geqslant 1$, $E_{j-1}(\mathcal{F}) \subseteq E_j(\mathcal{F})$.

Lemma 4. *Suppose (G,σ) is a signed graph and $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ is a family of optimal disjoint independent sets. Then for any $j \geqslant 0$, for any $e \in E_j(\mathcal{F})$, for any $F_i \in \mathcal{F}$, either $e \in F_i$ or $F_i + e$ contains a signed circuit.*

Proof. Assume the lemma is not true. Let t be the minimum integer for which the following holds:

There is a family of optimal disjoint independent sets $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$, an edge $e_t \in E_t(\mathcal{F})$ and an $F_{i_t} \in \mathcal{F}$ such that $e_t \notin F_{i_t}$ and $F_{i_t} + e_t$ contains no signed circuit.

First we observe that $t \neq 0$, for otherwise, by replacing F_{i_t} with $F_{i_t} + e_t$ in $\mathcal{F}$, we obtain a family of k disjoint independent sets, each contains a spanning tree of G, and their union contains one more edge. This is in contrary to the definition of a family of optimal disjoint independent sets. So $t \geqslant 1$ and $e_t \in E_t(\mathcal{F}) - E_{t-1}(\mathcal{F})$. By definition, there is an edge $e_{t-1} \in E_{t-1}(\mathcal{F}) - E_{t-2}(\mathcal{F})$ and an $F_{i_{t-1}} \in \mathcal{F}$ such that $e_t \in C(F_{i_{t-1}}, e_{t-1})$. We use the convention that $E_{-1}(\mathcal{F}) = \emptyset$.

We repeat the following procedure: Assume $1 \leqslant j \leqslant t$ and we have found $e_s \in E_s(\mathcal{F}) - E_{s-1}(\mathcal{F})$ for $j-1 \leqslant s \leqslant t$ so that $e_s \in C(F_{i_{s-1}}, e_{s-1})$ for $j \leqslant s \leqslant t$. If $F_{i_{j-1}} - e_j$ is connected or $j = 1$, then we stop. Otherwise, as $e_{j-1} \in E_{j-1}(\mathcal{F}) - E_{j-2}(\mathcal{F})$, there is an edge $e_{j-2} \in E_{j-2}(\mathcal{F}) - E_{j-3}(\mathcal{F})$ and an $F_{i_{j-2}} \in \mathcal{F}$ such that $e_{j-1} \in C(F_{i_{j-2}}, e_{j-2})$.

Assume the procedure above stops at q, i.e., either $F_{i_{q-1}} - e_q$ is connected or $q = 1$. For $i = 1, 2, \ldots, k$, let $I_i = \{q-1 \leqslant s \leqslant t\colon\ i_s = i\}$. Let

$$F'_i = F_i - \bigcup_{s \in I_i} e_{s+1} + \bigcup_{s \in I_i} e_s,$$

with the following convention: for any subgraph H, $H - e_{t+1} = H$ and if $q \geqslant 2$, then $H + e_{q-1} = H$. Note that e_{t+1} is not defined in our construction. If $q \geqslant 2$, the edge e_{q-1} does exist, however, instead of adding it to $F'_{i_{q-1}}$, this edge will stay at its original independent set. For example, if $2 \leqslant q = t$, i.e., $F_{i_{t-1}} - e_t$ is connected, then we have $F'_{i_t} = F_{i_t} + e_t$, $F'_{i_{t-1}} = F_{i_{t-1}} - e_t$ and $F'_j = F_j$ for $j \neq i_t, i_{t-1}$. If $2 \leqslant q = t-1$, then we have $F'_{i_t} = F_{i_t} + e_t$, $F'_{i_{t-1}} = F_{i_{t-1}} - e_t + e_{t-1}$, $F'_{i_{t-2}} = F_{i_{t-2}} - e_{t-1}$ and $F'_j = F_j$ for $j \neq i_t, i_{t-1}, i_{t-2}$. However, if $q = 1$, then the edge $e_{q-1} = e_0$ is added to F'_{i_0}. If $q = 1$, the edge e_0 is not contained in any of the F_i's.

Let $\mathcal{F}' = \{F'_1, F'_2, \dots, F'_k\}$. First we show that $\mathcal{F}'$ is a family of optimal disjoint independent sets. It suffices to show that each F'_i is a connected independent set.

Assume $I_i = \{j_1 > j_2 > \cdots > j_l\}$. Let $F_i^0 = F_i$, and for $s = 1, 2, \dots, l$, let $F_i^s = F_i^{s-1} - e_{j_s+1} + e_{j_s}$. We shall prove by induction that each F_i^s is a connected independent set. The case $s = 0$ follows from definition. Assume $s \geqslant 1$ and F_i^{s-1} is a connected independent set.

Assume first that $j_s \neq t$ and if $q \geqslant 2$ then $j_s \neq q$. By definition, $e_{j_s+1} \in C(F_i, e_{j_s})$. Recall that $C(F_i, e_{j_s})$ is either a balanced circuit or the disjoint union of two unbalanced circuits that are contained in a barbell of $F_i + e_{j_s}$. (Note that e_{j_s+1} and e_{j_s} are two distinct elements of $C(F_i, e_{j_s})$.) Since $F_i - e_{j_s+1}$ is disconnected, e_{j_s} and e_{j_s+1} are contained in a circuit of $C(F_i, e_{j_s})$. Now F_i^{s-1} is obtained from F_i by deleting and adding some edges e_j, where $j > j_s$. So the deleted and added edges e_j are not contained in $E_{j_s}(\mathcal{F})$. As all the edges of $C(F_i, e_{j_s})$ are contained in $E_{j_s}(\mathcal{F})$, so $C(F_i, e_{j_s})$ is contained in $F_i^{s-1} + e_{j_s}$. As F_i^{s-1} is connected and independent, $F_i^{s-1} + e_{j_s}$ contains at most one signed circuit. So $C(F_i, e_{j_s}) = C(F_i^{s-1}, e_{j_s})$. By the minimality of t, $F_i - e_{j_s+1}$ is disconnected, implying that e_{j_s+1} and e_{j_s} are contained in one circuit of $C(F_i, e_{j_s})$. Hence $F_i^s = F_i^{s-1} - e_{j_s+1} + e_{j_s}$ is connected and independent.

If $j_s = t$, then $F_i^s = F_i + e_{j_s}$, which is a connected independent set of G by the definition of t. If $q \geqslant 2$ and $j_s = q$, then $F_i^s = F_i^{s-1} - e_{j_s+1}$. By our construction procedure, $F_i - e_{j_s+1}$ is connected. By the argument in the previous paragraph, the only possibility for this to happen is that $C(F_i, e_{j_s})$ is the disjoint union of two unbalanced circuits that is contained in a barbel of $F_i + e_{j_s}$, and e_{j_s} and e_{j_s+1} are contained in different unbalanced circuits of $C(F_i, e_{j_s})$. Similarly as in the previous paragraph, we know that $C(F_i, e_{j_s})$ is contained in $F_i^{s-1} + e_{j_s}$. As F_i^{s-1} is connected and independent, we conclude that $C(F_i, e_{j_s})$ is also contained in a barbel of $F_i^{s-1} + e_{j_s}$. Therefore $F_i^s = F_i^{s-1} - e_{j_s+1}$ is a spanning tree of G, and hence a connected independent set.

Now we have proven that $\mathcal{F}'$ is a family of optimal disjoint independent sets. If $q = 1$, then $\mathcal{F}'$ contains one more edge than $\mathcal{F}$, contrary to the assumption that $\mathcal{F}$ is a family of optimal disjoint independent sets. Thus we have $q \geqslant 2$.

We claim that for $0 \leqslant h \leqslant q-1$, $E_h(\mathcal{F}) \subseteq E_h(\mathcal{F}')$. For $h = 0$, this follows directly from the definition. Assume $1 \leqslant h \leqslant q-1$ and $E_{h-1}(\mathcal{F}) \subseteq E_{h-1}(\mathcal{F}')$. To prove that $E_h(\mathcal{F}) \subseteq E_h(\mathcal{F}')$, it suffices to show that for any edge $e \in E_{h-1}(\mathcal{F})$, for each $1 \leqslant i \leqslant k$, $C(F_i, e) \subseteq C(F'_i, e)$. Although F'_i is obtained from F_i by deleting and adding some edges, however, the deleted edges are not contained in $E_h(\mathcal{F})$. On the other hand, $C(F_i, e) \subseteq E_h(\mathcal{F})$. Therefore $C(F_i, e) \subseteq F'_i + e$ and hence $C(F_i, e) \subseteq C(F'_i, e)$.

Now $e_{q-1} \in E_{q-1}(\mathcal{F}')$ and $F'_{i_{q-1}} + e_{q-1}$ contains no signed circuit. This is in contrary to the minimality of t. $\square$

Lemma 5. *If G is $2k$-edge-connected and (G, σ) is k-unbalanced, then $M(G, \sigma)$ has k disjoint connected bases.*

Proof. Let $\mathcal{F} = \{F_1, F_2, \dots, F_k\}$ be a family of optimal disjoint independent sets. If each F_i is a base, then we are done. Assume that F_1 is not a base, i.e., F_1 is a spanning tree of G. The other F_i's are either a spanning tree or a spanning tree with one edge added making a unique unbalanced circuit.

Let $E_j(\mathcal{F})$ be defined as above. As G is finite and $E_j(\mathcal{F}) \subseteq E_{j+1}(\mathcal{F})$, there is an index j^* such that $E_{j^*}(\mathcal{F}) = E_{j^*+1}(\mathcal{F})$.

Contract all the edges in $E_{j^*}(\mathcal{F})$, we obtain a graph H (parallel edges resulting from the contraction are retained, and loops are removed). If $e \in E_{j^*}$ is a contracted edge, then for any $F_i \in \mathcal{F}$, $F_i + e$ contains a signed circuit. This implies that $F_i + e$ contains a circuit C such that $e \in C$ and $C \subseteq E_{j^*}(\mathcal{F})$.

This means that F_i has a path connecting the two ends of e and this path is contracted. Denote by F_i' the graph obtained from F_i by contracting all the edges in $E_{j^*}(\mathcal{F})$. Since each contracted part is a connected subgraph of F_i, it follows that if F_i is a spanning tree of G, then F_i' is a spanning tree of H; if F_i is a spanning tree of G with one edge added, then F_i' is either a spanning tree (if the unique unbalanced circuit is contracted to a vertex) or a spanning tree of H with one edge added. In particular, F' has either $|V(H)|$ edges or $|V(H)|-1$ edges. If there is an $F_i \in \mathcal{F}$ which is not a base of $M(G,\sigma)$, then the total number of edges in $F_1', F_2', \ldots, F_k'$ is at most $k|V(H)|-1$.

If $|V(H)| \geqslant 2$, then since H is $2k$-edge-connected, each vertex in H has degree at least $2k$, and hence H has at least $k|V(H)|$ edges, which is a contradiction. Thus H has only one vertex. It follows that for any two points x, y of G, for any $F_i \in \mathcal{F}$, there is a path P in F_i connecting x and y and $E(P) \subseteq E_{j^*}(\mathcal{F})$. If e is an edge of F_i which is not contained in a circuit of F_i, then $e \in E_{j^*}(\mathcal{F})$. If $e \in E(F_i)$ is contained in a circuit C, then either $e \in E_{j^*}(\mathcal{F})$ or $C - e \subseteq E_{j^*}(\mathcal{F})$ or both. Therefore if F_i has a circuit, then it has at most one edge not in $E_{j^*}(\mathcal{F})$. Assume there are $t < k$ of the F_i's which contains a circuit. Then there are at most t edges of G not in $E_{j^*}(\mathcal{F})$. Let G' be obtained from G by removing all the edges of G not in $E_{j^*}(\mathcal{F})$. By our assumption, G is k-unbalanced. So G' is not balanced and contains an unbalanced circuit C. By Lemma 4, for each edge e of $C - F_1$, $F_1 + e$ contains a unique balanced circuit C_e. This is a contradiction, as the symmetric difference of $\{C_e\colon e \in C - F_1\} = C$ is unbalanced. $\square$

6. Graphs with high edge connectivity

The following lemma is proved in [12,2]:

Lemma 6. *A connected signed graph G has $\Phi_c(G) = 2$ if and only if G is eulerian and $\sigma(E(G)) = 1$.*

Lemma 7. *If a signed graph (G,σ) has two spanning subgraphs G_1 and G_2 such that each (G_i,σ) has a nowhere zero 2-flow and $E(G_1) \cup E(G_2) = E(G)$, then $\Phi_c(G,\sigma) \leqslant \Phi(G,\sigma) \leqslant 4$. If moreover, $E(G) - E(G_2)$ contains a base of $M(G,\sigma)$, then $\Phi_c(G,\sigma) < 4$.*

Proof. Assume G has two subgraphs G_1, G_2 such that each (G_i,σ) has a nowhere zero 2-flow and $E(G_1) \cup E(G_2) = E(G)$. Let τ be an orientation of (G,σ). Let ϕ_i be a nowhere zero 2-flow in (G_i,τ). Then $\phi_1 + 2\phi_2$ is a nowhere zero 4-flow of (G,τ).

For the moreover part, we assume that $E(G) - E(G_2)$ contains a base of $M(G,\sigma)$. By Lemma 3, for each $i = 1, 2$, there is an orientation τ_i of G_i such that for any signed subsets (X,θ) of $V(G_i)$, $|\partial^+_{G_i,\tau_i}(X,\theta)| = |\partial^-_{G_i,\tau_i}(X,\theta)|$.

Let τ be the orientation of G defined as

$$\tau(h) = \begin{cases} \tau_1(h), & \text{if } h \in H(G_1), \\ \tau_2(h), & \text{otherwise.} \end{cases}$$

Let X be an arbitrary signed subsets of $V(G)$. Let

$$\begin{aligned} a &= \left|\partial^+_{G_1,\tau_1}(X,\theta)\right| = \left|\partial^-_{G_1,\tau_1}(X,\theta)\right|, \\ b &= \left|\partial^+_{G_2,\tau_2}(X,\theta)\right| = \left|\partial^-_{G_2,\tau_2}(X,\theta)\right|, \\ c &= \left|\partial^+_{G_2,\tau_2}(X,\theta) \cap \partial_{G_1,\tau_1}(X,\theta)\right| \leqslant b, \\ d &= \left|\partial^-_{G_2,\tau_2}(X,\theta) \cap \partial_{G_1,\tau_1}(X,\theta)\right| \leqslant b. \end{aligned}$$

Then $|\partial^+_{G,\tau}(X,\theta)| = a + (b-c)$ and $|\partial^-_{G,\tau}(X,\theta)| = a + (b-d)$. So

$$\left|\partial^-_{G,\tau}(X,\theta)\right| - \left|\partial^+_{G,\tau}(X,\theta)\right| = c - d \leqslant c.$$

Since $E(G) - E(G_2)$ contains a base of $M(G,\sigma)$, by Lemma 2, for any signed set X with $\partial_{G,\sigma}(X,\theta) \neq \emptyset$, $\partial_{G,\sigma}(X,\theta) \cap (H(G) - H(G_2)) \neq \emptyset$. This implies that

$$\partial_{G_1,\tau_1}(X,\theta) \nsubseteq \partial_{G_2,\tau_2}(X,\theta).$$

Therefore

$$c < \left|\partial_{G_1,\tau_1}(X,\theta)\right| = 2a.$$

Therefore

$$\left|\partial^-_{G,\tau}(X,\theta)\right| < \left|\partial^+_{G,\tau}(X,\theta)\right| + 2a \leqslant 3\left|\partial^+_{G,\tau}(X,\theta)\right|. \quad (1)$$

By symmetry, we also have

$$\left|\partial^+_{G,\tau}(X,\theta)\right| < 3\left|\partial^-_{G,\tau}(X,\theta)\right|.$$

By Lemma 3, $\Phi_c(G) < 4$. □

Lemma 8. *Suppose (G,σ) is a s-bridgeless signed graph.*

1. *If $M(G,\sigma)$ has two disjoint connected bases, then $\Phi_c(G,\sigma) \leqslant \Phi(G,\sigma) \leqslant 4$.*
2. *If $M(G,\sigma)$ has three disjoint bases, two of them are connected, then $\Phi_c(G,\sigma) < 4$.*

Proof. We shall only consider the case that (G,σ) is unbalanced (the balanced case is proved similarly but easier).

By Lemma 7, to prove (1), it suffices to find two subgraphs G_1, G_2 such that each (G_i,σ) has $\Phi(G_i) = 2$. To prove (2), we need to find two subgraphs G_1, G_2 such that each (G_i,σ) has $\Phi(G_i) = 2$ and $E(G) - E(G_2)$ contains a base of $M(G,\sigma)$.

Assume (G,σ) has two disjoint connected bases F_1, F_2. As (G,σ) is unbalanced, each of F_i is a spanning tree with an edge added making a unique unbalanced circuit.

For each $e \in E(G) - F_i$, $F_i + e$ contains a signed circuit, which is either a balanced circuit C of (G,σ) or a barbell Q of (G,σ), consisting of two edge disjoint unbalanced circuits C_1, C_2 and a path P (possibly of length 0, in which case $V(C_1) \cap V(C_2) \neq \emptyset$) connecting the two circuits. In the former case, let $C_i(e) = C$, in the latter case, let $C_i(e) = C_1 \cup C_2$. In any case, $C_i(e)$ is an even subgraph (i.e., each vertex has even degree) with $\sigma(C_i(e)) = 1$.

Let $G_1 = \Delta_{e\in(E(G)-F_1)} C_1(e)$ and let $G_2 = \Delta_{e\in F_1} C_2(e)$. Then G_i $(i = 1,2)$ is connected (because F_{3-i} is contained in G_i) and $d_{G_i}(x)$ is even for $x \in V(G)$, $i = 1,2$ and $\sigma(G_i) = 1$. Therefore each (G_i,σ) has a nowhere zero 2-flow. In case (G,σ) has base F_3 disjoint from F_1 and F_2, then by the construction, we know that F_3 is contained in $E(G) - E(G_2)$. This completes the proof of Lemma 8. □

Theorem 5. *Suppose (G,σ) is a s-bridgeless signed graph. If G is 4-edge-connected, then $\Phi_c(G,\sigma) \leqslant \Phi(G,\sigma) \leqslant 4$. If G is a 6-edge-connected, then $\Phi_c(G,\sigma) < 4$.*

Proof. Assume G is 4-edge-connected and (G,σ) is s-bridgeless. Then (G,σ) is 2-unbalanced. By Lemma 5, (G,σ) has two disjoint connected bases. By Lemma 8, $\Phi_c(G,\sigma) \leqslant \Phi(G,\sigma) \leqslant 4$.

Assume G is 6-edge-connected and (G,σ) is signed bridgeless. If (G,σ) is 3-unbalanced, then by Lemma 5, (G,σ) has three disjoint connected bases. By Lemma 8, $\Phi_c(G,\sigma) < 4$.

Assume (G,σ) is not 3-unbalanced. Then (G,σ) is equivalent to a signed graph with exactly two negative edges. Without loss of generality, we assume that (G,σ) has exactly two negative edges e_1, e_2. Let G' be a graph obtained from G by adding an arbitrary negative edge e_3. Since G' is 6-edge-connected, and switching at vertices in a subset V' of $V(G')$ will change the sign of all the edges in the cut $E[V', \bar{V}']$ (the edges between V' and $\bar{V}'$), we conclude that any signed graph equivalent to G' contains at least 3 negative edges, i.e., G' is 3-unbalanced. By Lemma 5, $M(G',\sigma)$ has three disjoint bases F_1, F_2, F_3. Each F_i is a spanning subgraph with one edge added making a unique unbalanced circuit. Since (G',σ) has exactly three negative edges, each base contains exactly one of the three negative edges. Thus we may assume that $e_i \in F_i$ and $F'_i = F_i - \{e_i\}$ is a spanning tree of G. Note that $F'_3 + e_1$ is also a base of $M(G,\sigma)$.

For each $i = 1,2$, and for each $e \in (E(G) - F_i)$, let $C_i(e)$ be defined as in the proof of Lemma 8. Let $G_1 = \Delta_{e\in E(G)-F_1} C_1(e)$ and let $G_2 = \Delta_{e\in F'_1} C_2(e)$. Then G_i $(i = 1,2)$ is connected (because F'_{3-i} is contained in both G_i) and even and has $\sigma(G_i) = 1$. By Lemma 2, we have $\Phi_c(G_i) = 2$ for $i = 1, 2$. Moreover, the base $F_3' + e_1$ of $M(G, \sigma)$ is contained in $E(G) - E(G_2)$. We claim that $E(G_1) \cup E(G_2) = E(G)$. It is obvious that if $e \notin B_1$, then $e \in E(G_1)$, and if $e \in F_1'$, then $e \in E(G_2)$. Thus we only need to show that $e_1 \in E(G_1)$. For any edge $e \in E(G) - B_1$, if $e \neq e_2$, then $C_1(e)$ cannot contain e_1, for otherwise e_1 is the only negative edge of $C_1(e)$, in contrary to the fact that $\sigma(C_1(e)) = 1$. On the other hand, e_1 must be contained in $C_1(e_2)$, for otherwise e_2 is the only negative edge of $C_1(e_2)$, in contrary to the fact that $\sigma(C_1(e_2)) = 1$. Thus $e_1 \in \Delta_{e \in E(G) - F_1} C_1(e)$. By Lemma 7, we have $\Phi_c(G) < 4$. □

After the paper was accepted, we learned that Máčajová and Škoviera proved in [9] that eulerian s-bridgeless signed graphs (G, σ) have $\Phi(G, \sigma) \leqslant 4$. The proof is long. We give here a short proof of this result by using Theorem 5.

Corollary 2. *Suppose (G, σ) is an s-bridgeless signed graph and the underlying graph is eulerian, then $\Phi_c(G, \sigma) \leqslant \Phi(G, \sigma) \leqslant 4$.*

Proof. By Theorem 5, we only need to consider the case that G is not 4-edge connected. Assume the conclusion is not true and G is a counterexample with minimum number of vertices. It is shown in [2] that a signed graph (H, σ) admits a nowhere zero 2-flow if and only if H is eulerian and (H, σ) has an even number of negative edges. Thus we know that (G, σ) has an odd number of negative edges. As G is eulerian, each cut has even size, so we may assume that G has a 2-edge cut $B = \{e_1, e_2\}$. Let G_1, G_2 be the two components of $G - B$.

If G has two edge disjoint unbalanced circuits C_1, C_2, then let $\mathcal{C}$ be a circuit decomposition of $E(G)$ which contains C_1 and C_2. It is obvious that $\mathcal{C}$ contains two distinct circuits C_1', C_2' such that $G - C_i'$ is connected. If C_i' is balanced for some i, then $G - C_i'$ is eulerian and s-bridgeless (as $G - C_i'$ has two edge disjoint unbalanced circuits C_1, C_2, and unbalanced circuits remains unbalanced after a switch, and an eulerian signed graph has an s-bridge if and only if it is equivalent to a signed graph with a single negative edge). By the minimality of G, we know that $G - C_i'$ admits a nowhere zero 4-flow f. As C_i' itself admits a nowhere zero 2-flow g, the sum $f + g$ gives a nowhere zero 4-flow of G. Assume both C_1', C_2' are unbalanced, then for $i = 1, 2$, $G_i' = G - C_i'$ is an eulerian signed graph with an even number of negative edges and hence admits a nowhere zero 2-flow f_i. Then $f_1 + 2f_2$ is a nowhere zero 4-flow of G.

Assume that G does not have two edge disjoint unbalanced circuits. Then the two edges $\{e_1, e_2\}$ (which form a 2-edge cut) does not induce an unbalanced circuit, for otherwise, both G_1, G_2 are balanced and G is equivalent to an eulerian signed graph with a single negative edge, contrary to the assumption that G is s-bridgeless. Thus by switching at some vertices, if needed, we can assume that both e_1, e_2 are positive edges. Since G does not have two edge disjoint unbalanced circuits, at least one of G_1, G_2 is balanced. Observe that for any flow (not necessarily nowhere zero) f on a signed graph, if a cut separates a balanced part from the rest of the graph, the net flow through the cut is zero. So by assuming that both e_1, e_2 are oriented from G_1 to G_2, for any flow f on G, $f(e_1) + f(e_2) = 0$. We contract edge e_1, the resulting graph G' is eulerian. By the minimality of G, G' has a nowhere zero 4-flow f. Now f naturally induces a flow in G. As $f(e_1) = f(e_2)$, we conclude that f is a nowhere zero 4-flow on G. □

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